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A verification theorem in stochastic differential games with markovian switchings

Un teorema de verificación en juegos estocásticos diferenciales con cambio markoviano

ESCOBEDO-TRUJILLO, Beatris[1], COLORADO-GARRIDO, Darío[2], ALAFFITA-HERNÁNDEZ, Francisco[2] and HERRERA-ROMERO, José[1]

[1]*Facultad de Ingeniería, Universidad Veracruzana, Campus Coatzacoalcos. Av. Universidad Km 7.5, Col. Santa Isabel, C.P. 96355, Coatzacoalcos, Veracruz, México.*
[2]*Centro de Investigación en Recursos Energéticos y Sustentables. Universidad Veracruzana, Campus Coatzacoalcos. Av. Universidad Km 7.5, Col. Santa Isabel, C.P. 96355, Coatzacoalcos, Veracruz, México.*

ID 1st Author*: Beatris, Escobedo-Trujillo* / **ORC ID**: 0000-0002-8937-3019, **CVU CONACYT ID**: 173174

ID 1st Coauthor*: Darío, Colorado-Garrido* / **ORC ID**: 0000-0003-4157-1005, **CVU CONACYT ID**: 171579

ID 2nd Coauthor: *Francisco, Alaffita-Hernández* / **ORC ID**: 0000-0002-7971-6356

ID 3rd Coauthor: *José, Herrera-Romero* / **ORC ID**: 0000-0001-9462-0160, **CVU CONACYT ID**: 186700

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Abstract

The main objective in this paper is to solve the maximin problem in infinite-horizon zero-sum stochastic differential games with Markovian switchings. To this end, we propose a verification theorem which is proved using standard dynamic programming techniques. This theorem is applied to solve the maximin problem: maximize the expected utility from terminal wealth with the risk minimum in a financial market with Markovian switching assuming the mean rate of return of the stocks, is not given a priory (because risk).

Black-Scholes market, Hamilton-Jacobi-Bellman equations, zero-sum games

Resumen

El principal objetivo del trabajo consiste en encontrar la solución de un problema maximin dentro de un juego diferencial estocástico con cambios markovianos en horizonte infinito. Para tal fin, proponemos teorema de verificación el cual es probado usando técnicas de programación dinámica. El Teorema de verificación asegura la existencia de una solución al problema maximin: maximizar la utilidad esperada de la ganancia terminal con mínimo riesgo en un mercado financiero con cambios markovianos asumiendo que la tasa de rendimiento promedio de los activos es desconocida.

Mercado Black-Scholes, Ecuaciones de Hamilton-Jacobi-Bellman, Juegos de suma cero

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* Correspondence to Author (email: Ernestina.amguiano@itigual.edu.mx)
† Researcher contributing first author.

Introduction

In this work a verification theorem (Theorem 2.4) is proved which includes the Hamilton- Jacobi-Bellman (HJB) equation for two-player finite-horizon zero-sum stochastic differential games with Markovian switchings, see Escobedo et al. (2012) and Mao et al. (2006). This verification theorem allows to study the maximin (minimax) problem. As an application from theorem, a financial market in which the prices of assets are governed by stochastic differential equations with coefficients depending on a continuous-time finite state homogenous Markov chain is considered, see Baüerle et al. (2004) and Di Masi et al. (1994). The states of the Markov chain represent the market conditions.

The market is incomplete due to the randomness of the coefficients. To simplify the exposition we will suppose that the market consists of one bond and one risky stock only. Moreover, we consider a completely observable market, which means that the investor or decision-market has full knowledge of the different asset prices and the market conditions. *The problem in this application consists in finding investment portfolios that maximize the expected utility of terminal wealth (optimal investment problem).* This problem leads to a maximin problem whose solution we are interested to find. Using the verification theorem and assuming that the mean rate of return of the stock is not given a priori, the problem will be solved explicitly using logarithmic utility.

The work is organized as follows. In Section 2 we study the diffusion processes where the coefficients evolve as a continuous-time Markov chain. This kind of processes is called diffusion processes with Markovian switchings. Also, we study the general zero-sum stochastic differential game. In Section 3 the main theorem is proved. In Section 4 we describe the financial market we want to study and we show how to solve a optimal investment problem with the logarithmic utility using the verification Theorem 2.4.

1. Stochastic differential games with Markovian switchings

Let Y (t) be a continuous-time Markov chain on the filtered probability space $(\Omega, \mathrm{F}, \hat{\mathrm{F}}, \mathrm{P})$ which taking values in a finite space E = {1, 2, . . . , N } with generator $Q = \{q_{i,j}\}_{N\times N}$ given

$$P(Y(s+t)=j|Y(s)=i)=q_{ij}t+o(t), \tag{1}$$

for states $i \neq j$ the number $q_{ij} \geq 0$ is the transition rate from i to j, while $q_{ii} = -\Sigma_{j\neq i} q_{ij}$. We assume that Y (t) is right continuous with finite limits from the left. Consider the controlled Markov-modulated diffusion process (also known as a piecewise diffusion or a switching diffusion or a diffusion with Markovian switching) defined by

$$dX(t) = b\big(X(t), Y(t), u(t)\big)dt + \sigma\big(X(t), Y(t)\big)dW(t),$$

$$x(0) = x, \ \ Y(0) = i, \tag{2}$$

where $b : \mathbb{R}^n \times E \times U \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times E \rightarrow \mathbb{R}^{d\times n}$ are given functions, usually called the drift and the dispersion matrix, respectively, and W (·) is a n-dimensional standard Brownian motion independent of Y (·). The stochastic process u(·) is a U −valued process called the control process and the set $U \subset \mathbb{R}^m$ is called the control (or action) space.

To begin, we impose the following conditions on the coefficients b, σ and on the Markov chain Y (t).

Assumption 2.1. ***Ito conditions***. *The functions* b(x, i, u) *and* σ(x, i, u) *are measurables, and satisfy for some constant* K ≥ 0*:*

(a) *(Linear growth condition) for all* $x \in \mathbb{R}^d$ *and* i ∈ E
$|b(x, i, u) + \sigma(x, i, u)| \leq K(1 + |x|)$,
uniform u ∈ U.

(b) *(Local Lipschitz condition) for all* $x \in \mathbb{R}^d$,
$|b(x, i, u) - b(y, i, u)| + |\sigma(x, i, u) - \sigma(y, i, u)| \leq K\,|x - y|$,

(c) *In addition, the Markov chain* Y (t) *is such that the joint process* (X(t), Y (t)) *satisfies the Feller property.*
(d) *The control process* U *is a compact set.*

Assumption 2.1 (a)-(b) ensure the existence of a unique solution of (2). On the other hand, even though X(t) itself is not necessarily Markov, it is a well-known fact (see for instance Mao et al. (2006)) that the joint process (X(t),Y(t)) is Markov. Furthermore, Assumption 2.1 (c) implies that (X(t), Y (t)) is strong Markov.

Stochastic differential games

The main references in this section are, Escobedo et al. (2012), Oksendal (2003) and Mao et al. (2006). Consider the stochastic differential equation with Markovian switchings (2) and let f : $\mathbb{R}^d \times E \times U \to \mathbb{R}$ and g : $\mathbb{R}^d \times E \to \mathbb{R}$ be given functions, called the **payoff rate** and the **bequest function**, respectively. Let R be an open subset of $R^d \times E$, called the **solvency region**, and let

$$\tau_R := \inf\{t > 0: (X(t), Y(t)) \notin R\} \quad (3)$$

be the first exit time of (X(·), Y (·)) from R. We assume that

$$\mathbb{E}_z^u\left[\int_0^{\tau_R} \left|f(X(t), Y(t), u(t))\right| dt + \left|g(X(\tau_R), Y(\tau_R))\right|\right] < \infty \quad (4)$$

for all control process u(·) ∈ U and z ∈ R. Associated to each control u(·) and each initial point z = (x_0, i_0) ∈ R we define the **payoff functional** J^u(z) of the form

$$J^u(z) = \mathbb{E}_z^u\left[\int_0^{\tau_R} \left|f(X(t), Y(t), u(t))\right| dt + \left|g(X(\tau_R), Y(\tau_R))\right|\right] \quad (5)$$

Here and in (4) we interpret g(X(τ_R), Y (τ_R)) as 0 if $\tau_R = \infty$.

Now suppose that there are two controllers rather than one. Then the control u(t) has the form

$$u(t) = (\theta(t), \pi(t)), \qquad t \geq 0$$

where $\theta(\cdot)$ is a control (or strategy) of player 1, and $\pi(\cdot)$ is a control (or strategy) of player 2. In this case, the process X(·) in (2) denotes the state of a **two player stochastic differential game with Markovian switchings**.

Definition 2.2. *For the payoff functional* $J^{\theta(t),\pi(t)}(z)$ *in* (5) *with* u(t) = (θ(t), π(t)), *the game is called a* ***zero-sum game*** *if the payoff* $J^{\theta(t),\pi(t)}(z)$ *represents the gain for player* 1 *and the loss for player* 2.

For every initial state z = (x, i) ∈ R, player 1 tries to maximize $J^{\theta,\pi}$(z) over the set of her admissible controls θ(·), whereas player 2 tries to minimize $J^{\theta,\pi}$(z) over the set of her controls $\pi(\cdot)$. Let Θ and Π be given families of admissible controls θ and π, respectively. The functions

$$\mathcal{U}(z) := \inf_{\pi \in \Pi}\left(\sup_{\theta \in \Theta} J^{(\theta,\pi)}(z)\right) \quad (6)$$

$$\mathcal{L}(z) := \sup_{\pi \in \Theta}\left(\inf_{\theta \in \Pi} J^{(\theta,\pi)}(z)\right) \quad (7)$$

play an important role. The function $\mathcal{L}(z)$ is called the game's lower value, and $\mathcal{U}(z)$ is the game's upper value. Clearly, we have

$\mathcal{L}(z) \leq \mathcal{U}(z)$ for all $z \in$ R

If the upper and lower values coincide, then the game is said to have a **value**, and the **value of the game**, call it V(z), is the common value of $\mathcal{L}(z)$ and $\mathcal{U}(z)$, i.e.,

$$V(z) := \mathcal{L}(z) = \mathcal{U}(z) \text{ for all } z \in R \quad (8)$$

The general **maximin problem** consists, essentially, in finding a pair of strategies that attain the lower value (7). Hence, the precise definition of the problem we are interested in is as follows.

Problem 2.3. *Find* (θ*, π*) ∈ Θ × Π *such that*

$$\mathcal{L}(z) := \sup_{\pi \in \Theta}\left(\inf_{\theta \in \Pi} J^{(\theta,\pi)}(z)\right) = J^{(\theta^*,\pi^*)}(z) \forall\, z \in R \quad (9)$$

If such a pair (θ*, π*) *exists, then it is called a maximin strategy or an optimal control.*

Similarly, the **minimax problem** is to find (θ*, π*) ∈ Θ × Π such that

$$u(z) := \inf_{\pi\in\Pi}\left(\sup_{\theta\in\Theta} J^{(\theta,\pi)}(z)\right) = J^{(\theta^*,\pi*)}(z) \;\; \forall \;\; z \in \; R \qquad (10)$$

Strategies. We restrict ourselves to consider only admissible Markov controls in Problem 2.3. Hence we assume that

$$\theta(t) = \bar{\theta}\big(t, X(t), Y(t)\big) \text{ and } \pi(t) = \bar{\pi}\big(t, X(t), Y(t)\big), \qquad (11)$$

for some functions $\bar{\theta}$: $\mathbb{R}_+ \times \mathbb{R}^d \times E \to K_1$, $\bar{\pi}$: $\mathbb{R}_+ \times \mathbb{R}^d \times E \to K_2$, where K_1, K_2 are compact subsets of $U \subset \mathbb{R}^d$. As customary we do not distinguish notationally between θ and $\bar{\theta}$, π and π. Thus our controls can simply be identified with (deterministic) functions $\theta(t,z)$ and $\pi(t,z)$, $z \in \mathbb{R}^d \times$E.

Main Result

When the control $u = (\theta(t,z), \pi(t,z)) \in \Theta \times \Pi$ is Markovian, the corresponding system (2) becomes a diffusion process with Markovian switchings, whose generator $L^{\theta,\pi}$ is given by

$$L^{\theta,\pi}v(z) = v_x(z)b\big(z,\theta(t,z),\pi(t,z)\big) + \frac{1}{2}\,\mathrm{Trace}[v_{xx}(z)D(z,\theta(t,z),\pi(t,z)] + \sum_{j=1}^{N} qijv(z) \qquad (12)$$

witch $z = (x,i)$ in R, $D := \sigma\sigma^*$ and $v \in C^2$ $(\mathbb{R}^d \times E)$

Let $\mathfrak{T}$ be the set of all F_t–stopping times $\tau \le \tau_R$. We can now state the main result of this work:

Theorem 2.4. ***(The HJB equation for zero-sum differential games with Markovian switchings).*** *Suppose that* ν ∈ C^2(R) ∩ C($\bar{R}$) *satisfies the equation*

$$\sup_{\theta\in\Theta}\inf_{\pi\in\Pi}\{L^{\theta,\pi}v(z) + f(z,\theta,\pi)\} = 0 \;\forall\, z \in R \qquad (13)$$

with boundary condition

$$v\left(X^{\theta,\pi}(\tau_R), Y(\tau_R)\right) = g\left(X^{\theta,\pi}(\tau_R), Y(\tau_R)\right)\chi\{\tau_R < \infty\}$$

at time τ_R*. If* $\left(\hat{\theta}(t,z), \hat{\pi}(t,z)\right) \in K_1 \times K_2$ *are admissible Markov controls such that*

$$\sup_{\theta\in\Theta}\inf_{\pi\in\Pi}\{L^{\theta,\pi}v(z) + f(z,\theta,\pi)\} = 0 \;\forall\, z \in R$$
$$= L^{\hat{\theta},\hat{\pi}}v(z) + f(z,\hat{\theta},\hat{\pi}) = 0 \;\forall\, z \in R$$

then

$$v(z) = \mathcal{L}(z) = \sup_{\theta\in\Theta}\left(\inf_{\pi\in\Pi} J^{\theta,\pi}(z)\right) = \sup_{\theta\in\Theta} J^{\theta,\hat{\pi}}(z) = \inf_{\pi\in\Pi} J^{\hat{\theta},\pi}(z)) = J^{\hat{\theta},\hat{\pi}}(z), \qquad (14)$$

and
$(\hat{\theta}(t,z), \hat{\pi}(t,z))$ *is an optimal* (*Markov*) *control (15)*

Proof. Suppose that $v \in C^2(R) \cap C(\widehat{R})$ and $(\hat{\theta}(t,z), \hat{\pi}(t,z)) \in K_1 \times K_2$ satisfy the hypotheses of the theorem. Then we have

(i) $L^{\theta,\hat{\pi}(t,z)}v(z) + f\big(z,\theta,\hat{\pi}(t,z)\big) \le 0 \;\forall \in K_1, z \in R$

(ii) $L^{\hat{\theta}(t,z),\pi}v(z) + f\big(z,\hat{\theta}(t,z),\pi\big) \ge 0 \;\forall \pi \in K_2, z \in R$

(iii) $L^{\hat{\theta}(t,z),\hat{\pi}(t,z)}v(z) + f\left(z,\hat{\theta}(t,z),\hat{\pi}(t,z)\right) = 0 \;\forall z \in R$

(iv) $X^{\theta,\pi}(\tau_R) \in \partial R$ a.s. on $\{\tau_R > \infty\}$ and

$$\lim_{t\to\tau_R^-} v(X^{\theta,\pi}(t), Y(t)) = g(\tau_R), Y(\tau_R))\chi_{\{\tau_R>\infty\}}$$

a.s. for $(\theta,\pi) \in K_1 \times K_2$, $z \in R$.

(v) the family $\{v(X^{\theta,\pi}(\tau), Y(\tau))\}_{\tau\in\mathfrak{T}}$ is uniformly integrable, for all $z \in R$ and $(\theta,\pi) \in K_1 \times K_2$ because of the fact that v is in $C^2(R) \cap C(\bar{R})$.

Choose $(\theta,\pi) \in K_1 \times K_2$. Then by Dynkin's formula we have

$$\mathbb{E}^z[v(X(\tau_R^{(N)}), Y(\tau_R^{(N)}))] = v(x,i) + \mathbb{E}^z[\int_0^{\tau_R^{(N)}} L^{\theta,\pi}v\,(X(t),Y(t))dt], \qquad (16)$$

where $(X(t), Y(t)) = (X^{\theta,\pi}(t), Y(t))$ and

$\tau_R^{(N)} = \tau_R \wedge N \wedge \inf\{t > 0 : |X(t)| \ge N\}$, $N = 1,2,\ldots$

Our first goal is to show that $v(z) \le \mathcal{L}(z)$ for all z in R

(I) If we apply (16) to $(\hat{\theta},\hat{\pi})$ and use **(ii)** for all z we get $\mathbb{E}^z[v(X(\tau_R^{(N)}),Y(\tau_R^{(N)}))] \geq v(z) - \mathbb{E}^z[\int_0^{\tau_R^{(N)}} f(z,\hat{\theta}(t,X(t),Y(t)),\pi(t,X(t),Y(t)))\,dt]$,

or

$v(z) \leq \mathbb{E}^z[\int_0^{\tau_R^{(N)}} f(z,\hat{\theta}(t,X(t),Y(t)),\pi(t,X(t),Y(t)))\,dt + v(X(\tau_R^{(N)}),Y(\tau_R^{(N)}))]$.

Letting $N\to\infty$ and using **(iv)** and **(v)** we obtain

$$v(z) \leq J^{\hat{\theta},\pi}(z). \quad (17)$$

Since this holds for all $\pi \in \prod$ we deduce that

$$v(z) \leq \inf_{\pi\in\prod} J^{\hat{\theta},\pi}(z). \quad (18)$$

Hence

$$v(z) \leq \sup_{\theta\in\Theta}(\inf_{\pi\in\prod} J^{\hat{\theta},\pi}(x,i)) = \mathcal{L}(z)) \quad (19)$$

for all z in R.

Our second goal is to prove the reverse of (19), so that $\mathcal{L}(z) \leq v(z)$ for all z in R.

(II) If we apply (16) to $(\theta,\hat{\pi})$, with $\hat{\pi}\in\prod$, and use **(i)** for all z we get $\mathbb{E}^z[v(X(\tau_R^{(N)}),Y(\tau_R^{(N)}))] \geq v(z) - \mathbb{E}^z[\int_0^{\tau_R^{(N)}} f(z,\theta(t,X(t),Y(t)),\hat{\pi}(t,X(t),Y(t)))\,dt]$,

or $v(z) \geq \mathbb{E}^z[\int_0^{\tau_R^{(N)}} f(z,\theta(t,X(t),Y(t)),\hat{\pi}(t,X(t),Y(t)))\,dt + v(X(\tau_R^{(N)}),Y(\tau_R^{(N)}))]$.

Letting $N\to\infty$ and using **(iv)** and **(v)** we obtain

$$v(z) \geq J^{\theta,\hat{\pi}}(z) \geq \inf_{\pi\in\prod} J^{\theta,\pi}(z). \quad (20)$$

Since this holds for all $\theta\in\Theta$ we deduce that

$$v(z) \geq \sup_{\theta\in\Theta}(\inf_{\pi\in\prod} J^{\vartheta,\pi}(z) = \mathcal{L}(z). \quad (21)$$

(III) Finally, we apply (16) to $(\hat{\theta},\hat{\pi})$ and proceed as above. Then we end up with

$$v(z) = J^{\hat{\theta},\hat{\pi}}(z). \quad (22)$$

Combining (19), (21) and (22) we conclude that

$$\mathcal{L}(z) \leq v(z) = J^{\hat{\vartheta},\hat{\pi}}(z) \leq \mathcal{L}(z),$$

this proves (14) and (15) for all z in R. For our application, note that if $v\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^d\times E)$, then (2) becomes a diffusion process with Markovian switchings whose generator $L^{\theta,\pi}$ is given by

$$L^{\theta,\pi}v(t,z) = v_t(t,z) + v_x(t,z)b(t,z,\theta(t,z),\pi(t,z)) + \frac{1}{2}Trace[v_{xx}(t,z)D(t,z,\theta(t,z),\pi(t,z)] + \sum_{j=1}^{N} q_{ij}v(t,z). \quad (23)$$

for each $z\in\mathbb{R}^d\times E$.

Application: Financial market with Markovian switching

The main references in this section are, Baüerle et al. (2004), Di Masi et al. (1994), Fernholz (2002), Karatzas et al. (1998), Mataramvura et al. (2005).

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space endowed with a filtration $\hat{\mathcal{F}} = \{\mathcal{F}_t, t\geq o\}$, which is a nondecreasing right-continuous family of $\sigma-$algebras $\mathcal{F}_t$. We consider a *model of a financial market* as a pair of assets: a risk-free asset (bond) B and a risky (stock) asset S, which can be represented by their prices $B(t)$ and $S(t), t\in\mathbb{R}^+$. In this case one refers to a $(B,S)-$*market in continuous time*.

We assume that the prices of the bond and the stock are stochastic processes defined on the filtered probability space $(\Omega,\mathcal{F},\hat{\mathcal{F}},\mathcal{P})$. In the classical Black-Scholes formula for option pricing the price dynamics of the underlying asset is given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dW \;\; S(0) > 0, \quad (24)$$

the average rate of return μ and the volatility σ are constants. However, it has been proved by many authors that both of them, specially the volatility, are random processes in many situations. There is a strong evidence to indicate that the rate μ is a Markov jump process which can be modelled by a Markov chain. Of course, when the rate jumps, the volatility will jump accordingly. Taking these jumps into account, the classical model (24) has recently be generalized to form a new financial model

$$dS(t) = \mu(Y(t))S(t)dt + \sigma(Y(t))S(t)dW(t) \;\; S(0) > 0. \quad (25)$$

Here $Y(t)$ is a Markov chain with a finite state space $E = \{1,2,\cdot,N\}$and μ, σ are mapping from E to $[0,\infty)$. So, if the Markov chain is initially in state $Y(0) = i \in E_0$, then before its first jump from i_0 to $i_1 \in E$ at its first (random) jump timeτ_1, the underlying asset price obeys the following *geometric Brownian motion*

$$dS(t) = \mu(i_0)S(t)dt + \sigma(i_0)S(t)dW(t)$$

$$S(0) > 0 \qquad (26)$$

with initial value $X(t_0) = x_0$. During this period from t_0 to τ_1 the rate and volatility are $\mu(i_0)$ and $\sigma(i_0)$, respectively. At time τ_1, the Markov chain jumps at timeτ_2. During the period from $\tau_1$toτ_2, the underlying asset price obeys another geometric Brownian motion

$$dS(t) = \mu(i_1)S(t)dt + \sigma(i_1)S(t)dW(t)$$
$$S(0) > 0 \qquad (27)$$

with initial value $X(\tau_1)$ at time τ_1, and the rate and volatility have been switched to $\mu(i_1)$ and $\sigma(i_1)$ from $\mu(i_0)$ and $\sigma(i_0)$, respectively. The underlying asset price will continue to switch from one geometric Brownian motion to other according to the Markovian switching.

The equation (25) is known as the geometric Brownian motion with Markovian switching or the hybrid geometric Brownian motion, and a financial market is said to be financial market with Markovian switching if the stock price obeys the equation (25).

Definition 3.1. An stochastic process $\{X(t)\}_{0\le t\le T}$is said to be progressively measurable or progressive if for every$T \ge 0$, $\{X(t)\}_{0\le t\le T}$regarded as a function of $(t,\omega)$from $[0,T] \times \mathbb{R}^d$ is $B([0,T]) \times \mathcal{F}_t-$ measurable, where $B([0,T])$is the family of all Borel sub-sets of $[0,T]$.

Definition 3.2. A portfolio (or investment strategy) is a pair of $\mathcal{F}_t$–progressively measurable processes $\pi_0(t)$ and $\pi_1(t)$ that describe, respectively, the number of units of stock and of the bond that we hold at time t. The processes can take positive or negative values (we will allow unlimited short-selling of stock or bond). The value $V(t)$ of a portfolio $\pi(t) = (\pi_0(t), \pi_1(t))$ at time t is given by

$$V(t) = \pi_0(t)S(t) + \pi_1(t)B(t).$$

The definition of a (B, S)-market must be completed by indicating what kinds of portfolios can be used. The most important class consists of the self-financing portfolios.

Definition 3.3. A portfolio is self-financing if the change in its value depends on the change of the assets prices only. That is,

$$(\pi_0(t), \pi_1(t)) \ is\ self-financing$$
$$\leftrightarrow dV(t) = \pi_0(t)dS(t) + \pi_1(t)dB(t)$$

In the following we assume that the portfolio satisfies the self-financing property.

We consider a (B, S)-market in continuous time with Markovian switching. Moreover, in what follows, we suppose that $\mu, r, \sigma: E \to \mathbb{R}_+$ and $\mu(i) > r(i) > 0$ for all $i \in E$. Our model allows for random jumps of the interest rate r, the appreciation rate μ and the volatility σ. These jumps can be due to changes of external economic factors. The Markov chain $Y(\cdot)$ is used to represent the possible regimes of the financial environment.

Let $\pi(\cdot)$ be an admissible portfolio. The dynamics of the corresponding value process $V(t) = V^\pi(t)$ is

$$dV^\pi(t) = V^\pi(t)\pi_0(t)\frac{dS(t)}{S(t)} + V^\pi(t)(1 - \pi_0(t))\frac{dB(t)}{B(t)}$$
$$= V^\pi(t)\pi_0(t)[\mu(Y(t))dt + \sigma(Y(t))dW(t)] + V^\pi(t)(1-\pi_0(t))r(Y(t))dt$$
$$= V^\pi(t)[\{(1-\pi_0(t)r(Y(t)) + \pi_0(t)\mu(Y(t))\}dt + \pi_0(t)\sigma(Y(t))dW(t)], \quad (28)$$

with $V^\pi(0) > 0$, being the initial value, $\pi_0(t)$ is the fraction of the current wealth invested into the risky asset at time $t \in [0,T]$. The linear stochastic differential equation (28) can be solved explicitly and the solution is given by

$$V^\pi(t) = V^\pi(0)exp\left\{\int_0^t [rY(s)) + \pi_0(s)\mu(Y(s)) - r(Y(s)) + \frac{1}{2}\sigma^2(Y(s))\pi_0^2(s)]ds + \int_0^t \sigma(Y(s))\pi_0(s)dW(s)\right\}, \quad (29)$$

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assuming that

$$\int_0^T \{|1-\pi_0(t)r(Y(t))| + |\pi_0(t)\mu(Y(t))| + \pi_0^2(t)\sigma^2(Y(t))\}dt < \infty \ a.s.$$

Now, let us assume that the mean rate of return $\mu(Y(t))$ of the stock, is not given a priori, but it is the consequence of the portfolio choice $\pi(t)$ of a "representative" trader. The trader tries to maximize the expected utility of her terminal wealth by choosing her portfolio optimally, while the "market" tries to minimize this maximum expected utility by choosing μ (Y (t)) accordingly. This leads to the minimax problem.

$$\inf_{\mu\epsilon M}\left(\sup_{\pi\in\prod}\mathbb{E}\left[U_0\left(V^\pi(T)\right)\right]\right) \quad (30)$$

where U_0 is a given utility function and M is a given family of admissible processes μ (Y (t)). To put this problem in the framework of Section

2.2 y 2.3 we define the process X(t) = (X_0(t), X_1(t)) by

$$dX_0 = dt, X_0 = x_0 = s \in \mathbb{R} \quad (31)$$

and (see (28))

$$dX_1(t) = dV^\pi(t)\left[\{(1-\pi(t))r(Y(T)) + \pi(t)\mu(Y(t))\}dt + \pi(t)\sigma(Y(t))dW(t)\right], \quad (32)$$

With $X_1(0) = x_1 = x > 0$.

Then problem (30) can be formulated as follows:

Problem 3.4 *Find* $(\mu^*, \pi^*) \in M \times \prod$ *and* $\mathcal{U}(t,z)$ *such that*

$$\mathcal{U}(t,z) = \inf_{\mu\epsilon M}\left(\sup_{\pi\in\prod}\mathbb{E}\left[U_0\left(V^\pi(T)\right)\right]\right) = \mathbb{E}^{t,z}\left[U_0\left(V^{\mu^*,\pi^*}(T)\right)\right], \quad (33)$$

where z ∈ R *recall that* R *is the solvency region of the* process (X (·), Y (·)); see (3).
In this case, the generator $L^{\mu,\pi}$ of the process (32) has the form

$$L^{\mu,\pi}V(t,z) = \frac{\partial v(t,z)}{\partial t} + x[(1-\pi)r(i) + \pi\mu(i)]\frac{\partial v(t,z)}{\partial x} + \frac{1}{2}\sigma^2(i)\pi^2x^2\frac{\partial^2 v(t,z)}{\partial x^2} + \sum_{j\in E} q_{ij}v(t,x,j)$$

For $z = (x,i) \in R$ fixed and $v \in C^{1,2}(\mathbb{R}_+ \times R)$. The HJB equation associated to the Problem 3.4 can be written as

$$\begin{cases} \inf_{\mu\in K_1}\left(\sup_{\pi\in K_2}\{L^{\mu,\pi}v(t,x,i)\}\right) = 0 \ t < T \\ v(T,x,i) = U_0(x). \end{cases} \quad (34)$$

The following results are from [1].

Portfolio optimization with logarithmic utility. If we specify $U_0(x)$ to be of logarithmic form, i.e.,
$U_0(x) = \log(x)$,

then the value function $v(t,z) = v(t,x,i)$ in (34) is the form [1].

$$v(t,x,i) = \log(x) + g(t,i) \quad (35)$$

where $g(t,i)$ is the unique solution of the following system of linear differential equations

$$\frac{\partial g(t,i)}{\partial t} + r(i) + \frac{1}{2}\left(\frac{\mu(i)-r(i)}{\sigma(i)}\right)^2 + \sum_{j\in E} q_{ij}v(t,j) = 0, 0 \le t \le T$$

With boundary condition $g(T,i)$= 0 for $i \in E$. The process $\varphi: E \to \mathbb{R}_+$ defined as

$$\varphi(Y(t)) = \frac{\mu(Y(t)) - r(Y(t))}{\sigma(Y(t))},$$

with σ(Y(t)) >0 for all t ∈ [0, T], is called **the market price of risk.** Applying the generator $L^{\mu,\pi}$ to $v(t,x,i)$, we obtain

$$L^{\mu,\pi}v(t,x,i) = \frac{\partial g(t,i)}{\partial t} + \frac{1}{x}[(1-\pi)r(i) + \pi\mu] + \frac{1}{2}\pi^2\sigma^2(i)x^2\left(-\frac{1}{x^2}\right) + \sum_{j\in E} q_{ij}g(t,j) = \frac{\partial g(t,i)}{\partial t} + (1-\pi)r(i) + \pi\mu - \frac{1}{2}\pi^2\sigma^2(i) + \sum_{j\in E} q_{ij}g(t,j), \quad (36)$$

where $V^\pi(t) = x, \pi(t) = \pi$ *and* $\mu(i) = \mu$ *for* (t,x,i) *fixed*.

Maximizing (36) over π we obtain the following first-order condition for the maximum Point $\pi = \hat{\pi}$

$$\frac{\partial}{\partial \pi}L^{\mu,\pi}v(t,x,i) = -r(i) + \mu(i) - \pi\sigma^2(i) = 0$$

$$\hat{\pi} = \frac{\varphi(i)}{\sigma(i)} \quad (37)$$

Using the criterion of the second derivative we can see that the strategy $\hat{\pi}$ is really the maximum point because $\frac{\partial^2}{\partial\pi^2}L^{\mu,\pi}v(t,x,i) = -\sigma^2(i) < 0$.
Substituting $\hat{\pi}$ into (36), some tedious manipulations yield

$$L^{\mu,\hat{\pi}}v(t,x,i) = \frac{\partial g(t,i)}{\partial t} + r(i) + \frac{1}{2}\frac{r^2(i)}{\sigma^2(i)} + \frac{1}{2}\frac{\mu^2}{\sigma^2(i)} - \frac{r(i)}{\sigma^2(i)} + \sum_{j\in E} q_{i,j}g(t,i), \quad (38)$$

and minimizing over μ gives the first-order condition for the minimum point $\mu = \hat{\mu}$
$\frac{\partial}{\partial\mu}L^{\mu,\hat{\pi}}v(t,x,i) = \frac{\mu(i)}{\sigma^2(i)} - \frac{r(i)}{\sigma^2(i)} = 0$ obtaining

$$\hat{\mu} = r(i). \quad (39)$$

The criterion of the second derivative shows that μ is a minimum because
$\frac{\partial}{\partial\mu}L^{\mu,\hat{\pi}}v(t,x,i) = \frac{1}{\sigma^2(i)} > 0$.

Combining (37) with (39) and substituting in (36) we obtain

$$L^{\hat{\mu},\hat{\pi}}v(t,x,i) = \frac{\partial g(t,i)}{\partial t} + r(i) + \sum_{j\in E} q_{ij}g(t,j) = 0, \quad (40)$$

By substituting (40) into (34) we get the following problem:

Problem 3.5. Solve the linear system

$$\frac{\partial g(t,i)}{\partial t} + r(i) + \sum_{j\in E} q_{ij}g(t,j) = 0, \quad t < T, i \in E, \quad (41)$$

with boundary condition $v(T,x,i) = \log(x)$.

The Problem 3.5 was solved in [1] obtaining that

$$g(t,i) = \mathbb{E}^{t,x,i}\int_t^T \left[r\big(Y(s)\big) + \frac{1}{2}\varphi(Y(s))^2\right]ds. \quad (42)$$

Theorem 2.4 shows that the solution of Problem 3.4 in this case is:

$$\mathcal{U}(t,x,i) = v(t,x,i) = \log(x) + g(t,i).$$

with $g(t,i)$ given by (42). We conclude that in this game between the trader and the market, the market reacts to the trader's optimal portfolio choice by choosing

$$\hat{\mu}(t) = r\big(Y(t)\big), t \in [0,T].$$

Conclusions

Stochastic models with Markovian switching have recently been developed to model various financial quantities, such as option pricing and stock returns since there is a considerable interest, both from a theoretical and practical point of view, in a quantitative assessment of the risk involved in a financial position. By this reason, in this paper we show a form of evalued the financial risk using stochastic differential games theory.

Notation

a.s.: almost surely
$\mathbb{R}^d$: the d-dimensional Euclidean Space
$\mathbb{R}^{d\times n}$: the space of real d x n- matrices
σ^* : the transpose of a vector or matrix σ
v_x : the gradient of function v_x
v_{xx}: The Hessian matrix
$\chi_{\{M\}}$: characteristic funstion of set M
$\mathbb{E}^z$ [X]: expectation of the random variable X given the initial state z
C($\mathbb{R}^d$ x E): the space of real-valued continuous bounded functions on $\mathbb{R}^d$ x E
C^2 ($\mathbb{R}^d$ x E): The family of real-valued functions v defined on $\mathbb{R}^d$ x E which are twice continuously differentiable in x ∈ $\mathbb{R}^d$
$C^{1,2}$ ($\mathbb{R}_+$ x $\mathbb{R}^d$ x E): The family of all real-valued functions $v(t,x,i)$ defined on $\mathbb{R}_+$ x $\mathbb{R}^d$ x E which are twice continuously differentiable x∈ $\mathbb{R}^d$ and once continuously differentiable in t∈ $\mathbb{R}_+$

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